

# Extending $T^p$ automorphisms over $\mathbb{R}^{p+2}$ and realizing DE attractors

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## Abstract

We show that for any expanding map  $\phi : T^p \rightarrow T^p$ , there is an orientation-preserving self-diffeomorphism of  $\mathbb{R}^{p+2}$  realizing a hyperbolic attractor derived from  $\phi$ . The construction is based on a result in differential topology that for the standard unknotted embedding  $\iota_p : T^p \rightarrow \mathbb{R}^{p+2}$ , the subgroup  $E_{\iota_p}$  of  $\text{Aut}(T^p) \cong \text{SL}(p, \mathbb{Z})$  which consists of automorphisms that extend over  $\mathbb{R}^{p+2}$  as orientation-preserving diffeomorphisms, has index at most  $2^p - 1$ .

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## 1 Introduction

Except otherwise mentioned, in this paper all manifolds are smooth oriented, maps are smooth orientation-preserving. We use  $T^p$  to denote the standard  $p$ -torus, the product of  $p$  circles, and  $D^q$  to denote the (unit)  $q$ -disk in  $\mathbb{R}^q$ .

Hyperbolic attractors derived from expanding maps were introduced into dynamics by Smale as in his celebrated paper [Sm]. Smale posed four families of basic sets for his Spectral Decomposition Theorem on non-wandering set of diffeomorphisms on manifolds: *Group 0* which are zero dimensional ones such as isolated points and the Smale horseshoe; *Group A* and *Group DA*, both of which are derived from Anosov maps; and *Group DE* which are attractors derived from expanding maps. Note that any manifold admitting an expanding map is finitely covered by a Nil-manifold [Gr].

While the first three families arise easily or automatically from diffeomorphisms on manifolds, it is not known in general whether and how attractors of Group DE

could be realized via diffeomorphisms on manifolds (Definition 1.1 (3)). Originally DE attractors arose from self-embeddings of  $D^q$ -bundles over  $T^p$  (Definition 1.1 (2)), but such embeddings never extend to self-diffeomorphisms of the bundles for easy homotopy reasons. In this paper, we address the realization problem for the most basic case: realizing DE attractors derived from  $T^p$  by diffeomorphisms of  $\mathbb{R}^{p+q}$ . For this purpose we apply the following definitions.

**Definition 1.1.** (1) A real square matrix  $A$  is called *expanding*, if all of its eigenvalues are greater than 1 in absolute value. A linear map  $\phi : T^p \rightarrow T^p$  is expanding, if its induced matrix on  $H_1(T^p; \mathbb{Z})$  is expanding.

(2) A *hyperbolic bundle embedding*  $e : T^p \times D^q \rightarrow T^p \times D^q$  lifted from an expanding  $\phi : T^p \rightarrow T^p$  is an embedding which preserves the fibers, shrinks  $D^q$  factor evenly by some constant  $0 < \lambda < 1$ , and descends to  $\phi$  via the projection  $\pi : T^p \times D^q \rightarrow T^p$ .  $\Lambda = \bigcap_{i \geq 0} e^i(T^p \times D^q)$  is called a *type  $(p, q)$  attractor* derived from the expanding map  $\phi$ , or simply a *DE attractor*.

(3) Let  $f : M^{p+q} \rightarrow M^{p+q}$  be a diffeomorphism of a  $(p + q)$ -manifold. If there is an embedding  $T^p \times D^q \subset M$  such that  $\Lambda = \bigcap_{i \geq 0} f^i(T^p \times D^q)$  with  $\Lambda$  defined in (2), we say that  $f$  *realizes*  $\Lambda$ .

For example, the DE attractors given in [Sm] are of type  $(p, p + 1)$ , where the codimension  $q = p + 1$  is chosen as large as to guarantee the embeddings in Definition 1.1 (2).

Clearly if a DE attractor  $\Lambda$  is realized in  $\mathbb{R}^{p+q}$ , then it is realized in  $\mathbb{R}^{p+q'}$  for any larger codimension  $q' > q$ . There is no hyperbolic bundle embedding  $e : T^p \times D^1 \rightarrow T^p \times D^1$  lifted from an expanding  $\phi : T^p \rightarrow T^p$ ; in fact  $\Lambda$  in (2) is homeomorphic to a  $p$ -dimensional solenoid for any  $q$ , which cannot be embedded into  $(p + 1)$ -dimensional manifolds [JWZ]. Therefore  $q$  should be at least 2. We will prove in this paper that the codimension  $q = 2$  is enough in the following sense.

**Theorem 1.2.** *For any expanding map  $\phi : T^p \rightarrow T^p$ ,  $p \geq 1$ , there is a self-diffeomorphism of  $\mathbb{R}^{p+2}$  which realizes a hyperbolic attractor derived from  $\phi$ .*

*Remark 1.3.* A remarkable fact is that for global realization of DE attractors on closed manifolds the codimension  $q$  must be 2: if a diffeomorphism  $f$  on a closed, orientable  $n$ -manifold  $M$  with non-wandering set  $\Omega(f)$  a union of finitely many  $(\pm)$  DE attractors, then  $M$  is a rational homology sphere and each DE attractor is of type  $(n - 2, 2)$  [DPWY]. Such global realization problem is related to Smale's conjecture

that  $\Omega(f) = M$  for Anosov diffeomorphism  $f$  on a closed manifold  $M$ , and examples of such global realization can be found in [JNW] for  $n = 3$ .

While Theorem 1.2 is more in a dynamics character, its construction comes from a well understanding of the topological behavior of unknotted embeddings  $T^p \rightarrow \mathbb{R}^{p+2}$ , and unknotted, framing untwisted embeddings  $T^p \times D^2 \rightarrow \mathbb{R}^{p+2}$  (Definitions 2.2 and 3.3). More essence is involved when  $p \geq 2$  due to the existence of automorphisms which do not extend over  $\mathbb{R}^{p+2}$  diffeomorphically.

Let  $\iota : T^p \rightarrow \mathbb{R}^{p+2}$  be a smoothly embedded  $p$ -torus (with a fixed product structure) in  $\mathbb{R}^{p+2}$ . Define  $E_\iota$  to be the subgroup of  $\text{Aut}(T^p) \cong \text{SL}(p, \mathbb{Z})$  consisting of automorphisms that extend over  $\mathbb{R}^{p+2}$  as self-diffeomorphisms. The following theorem, which claims that  $E_\iota$  has finite index for unknotted embeddings, plays a key role in the proof of Theorem 1.2.

**Theorem 1.4.** *For the standard unknotted embedding  $\iota_p : T^p \rightarrow \mathbb{R}^{p+2}$  (Example 2.1),  $p \geq 1$ ,  $E_{\iota_p}$  is a subgroup of  $\text{Aut}(T^p)$  of index at most  $2^p - 1$ . More precisely, there is a subgroup of  $\text{Aut}(T^p)$ , identically the subgroup of  $\text{SL}(p, \mathbb{Z})$  of matrices whose entry sum of each column is odd, which has index  $2^p - 1$  and is contained in  $E_{\iota_p}$ .*

We say two unknotted embeddings are of the same *type* if they are equal up to a self-diffeomorphism of  $\mathbb{R}^{p+2}$  (Definition 2.2). It follows immediately that Theorem 1.4 also holds for standard type unknotted embeddings, namely, which are of the same type as the standard  $\iota_p$ . This might be a more convenient version since for any unknotted embedding, we have a standard type one with the same image (Proposition 2.4).

Using certain spin structure obstructions, one may show that for any unknotted embedding  $\iota : T^p \rightarrow \mathbb{R}^{p+2}$ ,  $p \geq 1$ ,  $E_\iota$  is a subgroup of  $\text{Aut}(T^p)$  of index at least  $2^p - 1$  [DLWY]. Therefore we have

**Corollary 1.5.** *For any standard type unknotted embedding  $\iota : T^p \rightarrow \mathbb{R}^{p+2}$ ,  $p \geq 1$ ,  $E_\iota$  is a subgroup of  $\text{Aut}(T^p)$  of index  $2^p - 1$ , as described in Theorem 1.4.*

*Remark 1.6.* The cosets of  $E_\iota$  in  $\text{Aut}(T^p)$  are in natural bijection with unknotted embeddings of ‘modular types’ (Definition 2.2), so Corollary 1.5 says that unknotted embeddings of  $T^p$  into  $\mathbb{R}^{p+2}$  have exactly  $2^p - 1$  modular types. It is worth pointing out that when  $p > 2$ , we do not know if there are non-modular types, basically because it is unknown whether the smooth mapping class group  $\pi_0 \text{Diff}_+(T^p)$  is isomorphic to  $\text{SL}(p, \mathbb{Z})$ , see Remark 2.3.

The case  $p = 1$  of Theorem 1.2 is in some sense well-known, for example, it is implicitly contained in [B] and [JNW]. The case  $p = 1$  of Theorem 1.4 is trivially true, and the case of  $p = 2$  is also known by [Mo] motivated by doing surgery along torus in 4-sphere. We will focus on  $p \geq 2$  in this paper.

The proof of Theorem 1.4 is contained in Section 2 and the proof of Theorem 1.2 is contained in Section 3. The proofs of Theorem 1.4 and Theorem 1.2 are motivated by several low dimension intuitions, so we will often explain the pictures before putting down formal proofs. An outline is as below.

In Section 2 we define unknotted embeddings  $\iota : T^p \rightarrow \mathbb{R}^{p+2}$ , describe their standard models and study their basic properties. For the standard model, we find three families (only two families for  $p = 2$ , also known by [Mo]) of elements of  $\text{Aut}(T^p)$  which extend over  $\mathbb{R}^{p+2}$  (Lemmas 2.8 and 2.9) and generate a subgroup  $G$  of index  $2^p - 1$  in  $\text{Aut}(T^p)$  (Lemma 2.10). This finishes the proof of Theorem 1.4. The full characterization of  $G$  (therefore of  $E_i$ ) and some technical results for Section 3 are also given in this section.

In Section 3, we first show that each expanding map  $\phi : T^p \rightarrow T^p$  can be lifted to a hyperbolic bundle embedding  $e : T^p \times D^2 \rightarrow T^p \times D^2$  (Proposition 3.1), possibly in many ways. Then we pick a ‘favorite’ lifting  $e$  (Example 3.2) in the sense that for any unknotted and framing untwisted embedding  $j : T^p \times D^2 \rightarrow \mathbb{R}^{p+2}$ , the composition  $j \circ e$  is still unknotted and framing untwisted (Lemmas 3.5, 3.6, 3.7 and 3.8). Finally applying the fact that  $E_i$  is of finite index in  $\text{Aut}(T^p)$  provided by Theorem 1.4, Theorem 1.2 is derived.

## 2 Unknotted $T^p$ and extendable automorphisms

In this section, we introduce and study unknotted embeddings of  $T^p$  into  $\mathbb{R}^{p+2}$ , and prove Theorem 1.4. We will regard  $S^1$  and  $D^2$  as the unit circle and disk of  $\mathbb{C}$ . The real and imaginary part of  $z \in \mathbb{C}$  are often written as  $z_x, z_y$ . The standard basis of  $\mathbb{R}^n$  is  $(\vec{\varepsilon}_1, \dots, \vec{\varepsilon}_n)$ , and the  $m$ -subspace spanned by  $(\vec{\varepsilon}_{i_1}, \dots, \vec{\varepsilon}_{i_m})$  will be written as  $\mathbb{R}_{i_1, \dots, i_m}^m$ . Note there is a natural inclusion of  $\mathbb{R}^n = \mathbb{R}_{1, \dots, n}^n$  into  $\mathbb{R}^{n+1}$ .

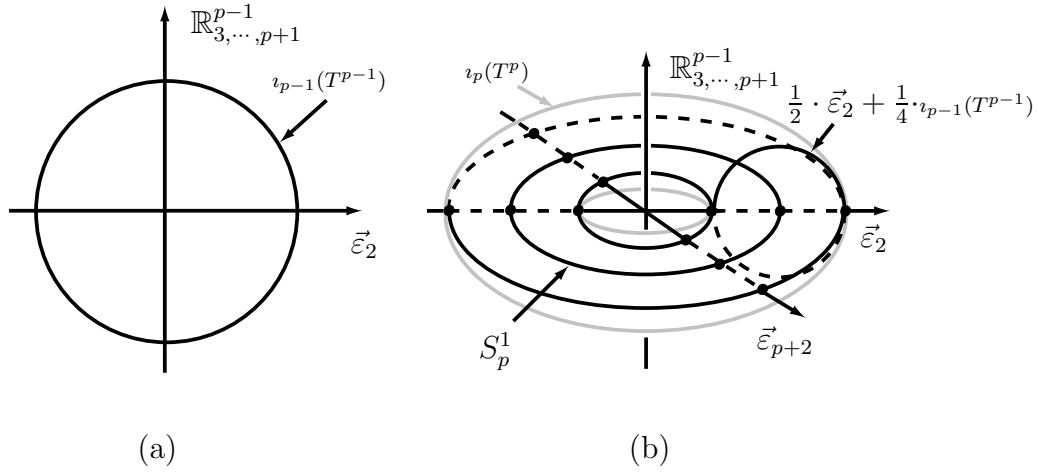
We start by investigating some important aspects of unknotted embeddings. It is reasonable to expect that such embeddings are fairly simple and symmetric, and agree much with our low-dimension intuition.

**Example 2.1** (The standard model). Let  $\iota_0 : \text{pt} = T^0 \rightarrow \mathbb{R}^2$  be  $\iota_0(\text{pt}) = 0$  by

convention. Inductively suppose  $\iota_{p-1}$  is constructed ( $p \geq 1$ ) such that  $\iota_{p-1}(T^{p-1}) \subset \text{Int}(D^p) \subset \mathbb{R}_{2,\dots,p+1}^p$ , see Figure 1 (a). Denote the rotation of  $\mathbb{R}^{p+2}$  on the subspace  $\mathbb{R}_{2,p+2}^2$  of angle  $\arg(u)$  as  $\rho_p(u) \in \text{SO}(p+2)$ , for any  $u \in S^1$ , and we may define  $\iota_p : T^p = T^{p-1} \times S_p^1$  as

$$\iota_p(v, u) = \rho_p(u) \left( \frac{1}{2} \cdot \vec{\varepsilon}_2 + \frac{1}{4} \cdot \iota_{p-1}(v) \right).$$

This explicitly describes an embedding of  $T^p = S_1^1 \times \dots \times S_p^1$  into  $\mathbb{R}_{2,\dots,p+2}^{p+1}$ , see Figure 1 (b), where  $T^p \subset \mathbb{R}_{2,\dots,p+2}^{p+1}$  is presented, and  $\vec{\varepsilon}_1$  is perpendicular to the page. Observe that the image of  $T^p$  is invariant under  $\rho_p(u)$ .



**Definition 2.2.** An embedding  $\iota : T^p \rightarrow \mathbb{R}^{p+2}$  is called *unknotted* if there is a diffeomorphism  $g : \mathbb{R}^{p+2} \rightarrow \mathbb{R}^{p+2}$  such that  $\iota$  and  $g \circ \iota_p$  have the same image, i.e.  $\iota(T^p) = g \circ \iota_p(T^p)$ . Unknotted embeddings  $\iota_0, \iota_1 : T^p \rightarrow \mathbb{R}^{p+2}$  are called of the same *type* if they are the same up to a  $\mathbb{R}^{p+2}$  self-diffeomorphism, namely there is a diffeomorphism  $h : \mathbb{R}^{p+2} \rightarrow \mathbb{R}^{p+2}$  such that  $h \circ \iota_0 = \iota_1$ . This is an equivalence relation, and the equivalent classes are called *types*. The type of  $\iota_p$  is called the *standard type*.

For any  $\tau \in \text{Aut}(T^p)$ ,  $\tau$  defines a *modular transformation* on types, namely  $[i] \mapsto [i \circ \tau]$ . A *modular type* is obtained by a modular transformation of the standard type.

*Remark 2.3.* Clearly modular types are in one-to-one correspondence with the cosets in  $\text{Aut}(T^p)/E_{\iota_p}$ . Note we do not know if there are infinitely many types, mainly because we do not know if the smooth mapping class group  $\pi_0 \text{Diff}_+(T^p) \cong \text{SL}(p, \mathbb{Z})$ . Fortunately, it will be sufficient for our purpose to consider only modular types, which are indeed finitely many (Theorem 1.4).

**Proposition 2.4.** *For any unknotted embedding and any type, there is an unknotted embedding with the same image and of that type.*

*Proof.* Let  $\iota_0$  be the embedding, and  $[\iota_1]$  be the type. By Definition 2.2, there is some  $\mathbb{R}^{p+2}$ -self-diffeomorphism  $h_1$  such that  $h_1 \circ \iota_0(T^p) = \iota_1(T^p)$ . Let  $\tau = \iota_1^{-1} \circ h_1 \circ \iota_0 : T^p \rightarrow T^p$ , then  $h_1^{-1} \circ \iota_1 = \iota_0 \circ \tau^{-1}$ . Thus  $\iota_0 \circ \tau^{-1}$  has the same image as  $\iota_0$ , and the same type as  $[\iota_1]$ .  $\square$

We need to understand unknotted embeddings by taking a closer look at the standard model. Let  $B^4$  denote the 4-dimensional disk centered at the origin with radius 2.

**Lemma 2.5.** *In the standard model for  $p \geq 2$ , for each  $i = 2, \dots, p$ , the embedding  $\iota_p : T^p = S_1^1 \times \dots \times S_p^1 \rightarrow \mathbb{R}^{p+2}$  extends as an embedding*

$$k_{1i} : B^4 \times (S_2^1 \times \dots \times \hat{S}_i^1 \times \dots \times S_p^1) \rightarrow \mathbb{R}^{p+2},$$

where  $S_1^1 \times \dots \times S_p^1 \subset B^4 \times (S_2^1 \times \dots \times \hat{S}_i^1 \times \dots \times S_p^1)$  is the standard embedding (i.e. inclusion) for  $S_1^1 \times S_i^1 \subset B^4$ , and the identity on other factors.

*Proof.* To see the idea consider a standard  $T^2 = S_1^1 \times S_2^1$  in  $\mathbb{R}^4$ . Make a solid torus  $D_1^2 \times S_2^1$  by filling up the  $S_1^1$  factor, and attach a semi-sphere in  $\mathbb{R}^4$  along its core. The result is a ‘hat’ whose regular neighborhood is diffeomorphic to  $D^4$ . When  $i > 2$ , one only cares about  $S_1^1$  and  $S_i^1$ .

Inductively we first extend  $\iota_1$  as  $j_1 : D_1^2 \rightarrow \mathbb{C} \cong \mathbb{R}_{2,3}^2$ ; and suppose  $j_{s-1} : D_1^2 \times S_2^1 \times \dots \times S_{s-1}^1 \rightarrow \text{Int}(D^s) \subset \mathbb{R}_{2,\dots,s+1}^s$  is constructed, then define  $j_s : (D_1^2 \times S_2^1 \times \dots \times S_{s-1}^1) \times S_s^1 \rightarrow \text{Int}(D^{s+1}) \subset \mathbb{R}_{2,\dots,s+2}^{s+1}$  as

$$j_s(v, u) = \rho_s(u) \left( \frac{1}{2} \cdot \vec{\varepsilon}_2 + \frac{1}{4} \cdot j_{s-1}(v) \right).$$

After  $i - 1$  steps we obtain  $j_i : D_1^2 \times S_2^1 \times \dots \times S_i^1 \rightarrow \text{Int}(D^{i+1}) \subset \mathbb{R}_{2,\dots,i+2}^{i+1}$ . Now let  $\zeta_s(re^{i\theta})$  ( $0 \leq r \leq 1$ ) be the rotation of  $\mathbb{R}^{s+2}$  of angle  $\arccos(r)$ , on the subspace spanned by  $\rho_s(e^{i\theta})(\vec{\varepsilon}_2)$  and  $\vec{\varepsilon}_1$  (from the former toward the latter). We may further define  $k_{1i,i} : (D_1^2 \times S_2^1 \times \dots \times S_{i-1}^1) \times D_i^2 \rightarrow \text{Int}(D^{i+2}) \subset \mathbb{R}^{i+2}$  as, for example,

$$k_{1i,i}(v, re^{i\theta}) = \zeta_i \left( \frac{2r}{1+r^2} e^{i\theta} \right) (j_i(v, e^{i\theta})).$$

Then repeat the standard construction, namely, let  $k_{1i,s}(\vec{x}, u) = \rho_s(u)(\frac{1}{2} \cdot \vec{\varepsilon}_2 + \frac{1}{4} \cdot k_{1i,s-1}(\vec{x}))$  for  $i < s \leq p$ . In the end we obtain

$$k_{1i} = k_{1i,p} : D_1^2 \times D_i^2 \times (S_2^1 \times \cdots \times \hat{S}_i^1 \times \cdots \times S_p^1) \rightarrow \mathbb{R}^{p+2}.$$

From the construction, we see that  $k_{1i}$  can be extended a bit as an embedding

$$k_{1i} : B^4 \times (S_2^1 \times \cdots \times \hat{S}_i^1 \times \cdots \times S_p^1) \rightarrow \mathbb{R}^{p+2}.$$

□

**Corollary 2.6.** *For each  $i = 1, \dots, p$ ,  $\iota_p$  also extends as an embedding*

$$k_i : D^3 \times (S_1^1 \times \cdots \times \hat{S}_i^1 \times \cdots \times S_p^1) \rightarrow \mathbb{R}^{p+2},$$

where  $S_1^1 \times \cdots \times S_p^1 \subset D^3 \times (S_1^1 \times \cdots \times \hat{S}_i^1 \times \cdots \times S_p^1)$  is an unknotted embedding for  $S_i^1$  in  $\text{Int}(D^3)$ , and the identity on other factors.

*Proof.* Clearly the inclusion  $S_1^1 \times S_i^1 \subset B^4 \subset \mathbb{C} \times \mathbb{C}$  in Lemma 2.5 extends as an embedding  $S^1 \times D^3 \cong (S_1^1 \times D^1) \times B^2 \subset B^4 \subset \mathbb{C} \times \mathbb{C}$  where  $S_1^1 \times D^1$  is a tubular neighborhood of  $S_1^1 \subset \mathbb{C}$  and  $B^2$  is the 2-dimensional disk centered at the origin with radius  $\frac{3}{2}$ . Now  $k_i$  may be defined as  $k_{1i}$  composed with the latter embedding. □

Now let us consider  $E_i$  of the standard unknotted embedding  $\iota = \iota_p : T^p = S_1^1 \times \cdots \times S_p^1 \rightarrow \mathbb{R}^{p+2}$ . With the product structure of  $T^p$ ,  $\text{Aut}(T^p)$  is identified with  $\text{SL}(p, \mathbb{Z})$  ( $p \geq 2$ ). Denote

$$R_{ij} = I + E_{ij}, \quad Q_{ij} = R_{ij}^{-1} R_{ji} R_{ij}^{-1},$$

where  $i \neq j$ , and  $I$  is the identity matrix and  $E_{ij}$  has 1 for the  $(i, j)$ -entry and all other entries 0. Note that  $R_{ij}$  is the full Dehn twist on the sub-torus  $S_i^1 \times S_j^1$  along  $S_i^1$ , and  $Q_{ij}$  trades the two factors of  $S_i^1 \times S_j^1$ .

Call a twice full Dehn twist along the circle  $c$  on the torus  $T^2$  the *Dehn 2-twist* along  $c$ . When  $p = 2$ , there are two basic extendable automorphisms for the embedding  $T^2 = S_1^1 \times S_2^1 \subset \mathbb{C} \times \mathbb{C} \cong \mathbb{R}^4$ .

**Lemma 2.7.** *For the standard embedding  $T^2 = S_1^1 \times S_2^1 \subset B^4 \subset \mathbb{C} \times \mathbb{C} \cong \mathbb{R}^4$ , the following automorphisms can be extended as  $B^4$  self-diffeomorphisms with the identity near the boundary:*

- (1) the Dehn 2-twist along each factor circle;
- (2) trading two factors with the orientation preserved.

*Proof.* (1) It suffices to prove for the first factor. Consider  $S_1^1 \times S_2^1 \subset (S_1^1 \times D^1) \times D^2 = S_1^1 \times D^3 \subset \mathbb{R}^4$ , where  $S_1^1 \times D^1$  is a tubular neighborhood of  $S_1^1$  in the first  $\mathbb{C}$ , and  $D^2$  is the disk bounded by  $S_2^1$  in the second  $\mathbb{C}$ ,  $S_1^1 \times D^3$  is a tubular neighborhood of  $S_1^1$  in  $\mathbb{R}^4$ ,  $\partial(S_1^1 \times D^3) = S_1^1 \times S^2$ , and  $* \times S_2^1$  is the equator of  $* \times S^2$ .

The Dehn 2-twist  $\tau : S_1^1 \times S_2^1 \rightarrow S_1^1 \times S_2^1$  is  $(x, y) \mapsto (x, x^2 y)$ . The map  $x \mapsto x^2$ , considered as a map from  $S^1$  to  $\text{SO}(2)$ , is of degree 2. Thus the map  $x \mapsto x^2$ , considered as a map  $g : S^1 \rightarrow \text{SO}(3)$ , is homotopic to a constant map since  $\pi_1(\text{SO}(3)) \cong \mathbb{Z}_2$ . We may extend the map  $\tau : S_1^1 \times S_2^1 \rightarrow S_1^1 \times S_2^1$  to a map  $\tilde{\tau} : S_1^1 \times S^2 \rightarrow S_1^1 \times S^2$ , defined by  $\tau(x, y) = (x, g(x)y)$ . Since  $g : S^1 \rightarrow \text{SO}(3)$  is homotopic to a constant map,  $\tilde{\tau}$  is diffeotopic to the identity. By the isotopy extension theorem,  $\tilde{\tau}$  can be extended to a diffeomorphism of  $B^4$  identity near the boundary.

(2) First extend  $T^2 = S_1^1 \times S_2^1 \rightarrow S_1^1 \times S_2^1$  as  $f : \mathbb{R}^4 = \mathbb{C} \times \mathbb{C} \rightarrow \mathbb{C} \times \mathbb{C}$ , by  $(z, w) \mapsto (\bar{w}, z)$ .  $f$  is an orientation-reversing diffeomorphism. To adjust to get an orientation-preserving one, pick a self-diffeomorphism  $h$ , being the identity outside a compact subset of  $\text{Int}(B^4)$ , such that  $h(T^2)$  lies on the subspace  $\mathbb{R}_{2,3,4}^3$ . Let  $r_1 : \mathbb{R}^4 \rightarrow \mathbb{R}^4$  be the reflection with respect to  $\mathbb{R}_{2,3,4}^3$ , then  $r_1$  is orientation-reversing, and is the identity restricted to  $h(T^2)$ . Now  $f_1 = (h^{-1}r_1h) \circ f$  is orientation preserving, extending the described automorphism on  $T^2$ . Furthermore, since  $f_1(z, w) = (-w, z)$  when  $|w|^2 + |z|^2 > 4 - \varepsilon$ , where  $\varepsilon > 0$  is sufficiently small, we may adjust  $f_1$  near the boundary of  $B^4$  to get an  $f_2$  which is the identity near the boundary of  $B^4$ .  $\square$

From this observation we have the following lemma for  $p \geq 2$ .

**Lemma 2.8.**  $R_{1i}^2, Q_{1i} \in E_i$ , for  $1 < i \leq p$ .

*Proof.* Let  $\tau$  be either  $R_{1i}^2$  or  $Q_{1i}$ . From Lemma 2.7,  $\tau|_{S_1^1 \times S_i^1} : S_1^1 \times S_i^1 \rightarrow S_1^1 \times S_i^1$  extends as  $\bar{\tau} : B^4 \rightarrow B^4$  which is the identity near the boundary. Therefore by Lemma 2.5,

$$\bar{\tau} \times \text{id} : B^4 \times (S_2^1 \times \cdots \times \hat{S}_i^1 \times \cdots \times S_p^1) \rightarrow B^4 \times (S_2^1 \times \cdots \times \hat{S}_i^1 \times \cdots \times S_p^1)$$

induces a self-diffeomorphism of  $k_{1i}(B^4 \times (S_2^1 \times \cdots \times \hat{S}_i^1 \times \cdots \times S_p^1))$  which is the identity near the boundary. The latter further extends to a diffeomorphism of  $\mathbb{R}^{p+2}$  by the identity outside the image of  $k_{1i}$ .  $\square$

However, these automorphisms are not quite enough for generating  $E_i$  when  $p \geq 3$ . Extra extendable ones come from the following geometric construction.

**Lemma 2.9.**  $R_{1p}R_{ip} \in E_i$ , for  $1 < i < p$ .



*Proof.* According to Lemma 2.5, we will first extend  $\eta = R_{1p}R_{ip}$  on  $T^p$  over  $B^4 \times (S_2^1 \times \cdots \times \hat{S}_i^1 \times \cdots \times S_p^1)$ , identity near the boundary, then extend it further as a self-diffeomorphism of  $\mathbb{R}^{p+2}$  via  $k_{1i}$  by the identity outside the image. Since  $\eta = \tau \times \text{id}$  as from  $(S_1^1 \times S_i^1 \times S_p^1) \times (S_2^1 \times \cdots \times \hat{S}_i^1 \times \cdots \times S_{p-1}^1)$  to itself, essentially one must extend

$$\tau : S_1^1 \times S_i^1 \times S_p^1 \rightarrow S_1^1 \times S_i^1 \times S_p^1$$

as  $B^4 \times S_p^1 \rightarrow B^4 \times S_p^1$ . It is easy to see the matrix of  $\tau$  is  $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ , the

$(1, i, p)$  minor of  $R_{1p}R_{ip}$ . It follows that each column sum of  $R_{1p}R_{ip}$  is odd. For conciseness rewrite  $u_p$  as  $w$ , then we have  $\tau((u_1, u_i), w) = (\mu_w(u_1, u_i), w)$ , where  $\mu_w(u_1, u_i) = (wu_1, wu_i)$  is an action of  $S_p^1$  on  $T^2$  by flowing along the diagonal-loop direction.

We extend  $\tau$  as follows. First, via the inclusion  $T^2 = S_1^1 \times S_i^1 \subset S^3$  (in fact, the sphere centered at the origin with radius  $\sqrt{2}$ ), the diagonal-loop fibration on  $T^2$  extends as the Hopf fibration on  $S^3$ , and  $\mu_w$  also extends as  $\tilde{\mu}_w : S^3 \rightarrow S^3$  by flowing along the fiber loops. Thus  $\tau$  extends as  $\tilde{\tau} : S^3 \times S_p^1 \rightarrow S^3 \times S_p^1$ ,  $\tilde{\tau}(x, w) = (\tilde{\mu}_w(x), w)$ .

On the other hand,  $\tilde{\mu}_w$  can be regarded as  $S_p^1 \rightarrow \text{Diff}_+(S^3)$  which is given by a Lie group left multiplication, regarding  $S_p^1$  as a subgroup of  $S^3$ . Thus  $\tilde{\mu}_w$  extends as  $S^3 \rightarrow \text{Diff}_+(S^3)$  by the Lie group left multiplication. This implies that  $\tilde{\mu}_w$  is homotopic to the constant identity in  $\pi_1 \text{Diff}_+(S^3)$  (since any circle is null-homotopic in the 3-sphere). Thus  $\tilde{\tau}$  is diffeotopic to the identity. By the isotopy extension theorem,  $\tilde{\tau}$  can be extended to a diffeomorphism  $\bar{\tau}$  of  $B^4 \times S_p^1$ , which is the identity near the boundary.

Now  $\bar{\eta} = \bar{\tau} \times \text{id}$  is a self-diffeomorphism of  $(B^4 \times S_p^1) \times (S_2^1 \times \cdots \times \hat{S}_i^1 \times \cdots \times S_{p-1}^1)$ , or diffeomorphically, of  $B^4 \times (S_2^1 \times \cdots \times \hat{S}_i^1 \times \cdots \times S_p^1)$ , being the identity near the boundary. Finally extend  $\bar{\eta}$  to a diffeomorphism  $\mathbb{R}^{p+2} \rightarrow \mathbb{R}^{p+2}$  via  $k_{1i}$  with the identity outside the image.  $\square$

Here we need an elementary matrix lemma, and the technical result (2) is in order for Section 3.

**Lemma 2.10.** (1) *The subgroup  $G$  of  $\text{SL}(p, \mathbb{Z})$  generated by  $R_{1i}^2$ ,  $Q_{1i}$  ( $1 < i \leq p$ ) and  $R_{1p}R_{ip}$  ( $1 < i < p$ ) is of index  $2^p - 1$ . More precisely, it consists of all the matrices  $U \in \text{SL}(p, \mathbb{Z})$  whose entry sum of each column is odd.*

(2) For any  $1 \leq i \leq p$ , any  $U \in \text{SL}(p, \mathbb{Z})$  can be written as  $KJ$  such that  $K$  is a word in  $R_{1j}^2$ ,  $Q_{1j}$ ,  $1 < j \leq p$ , and  $J$  has the minor  $J_{ii}^* = 1$ .

*Proof.* (1) Because  $R_{ij}^2 = Q_{1i}R_{1j}^2Q_{1i}^{-1}$ ,  $Q_{ij} = Q_{1i}Q_{1j}Q_{1i}^{-1}$ , for  $1 < i \neq j \leq p$ , we have  $R_{ij}^2, Q_{ij} \in G$  ( $i \neq j$ ). Also  $R_{1k}R_{jk} = Q_{kp}^{-1}R_{1p}R_{jp}Q_{kp}$  ( $k \neq 1, j, p$ ), and  $R_{ik}R_{jk} = Q_{1i}R_{1k}R_{jk}Q_{1i}^{-1}$  ( $i \neq 1, j, k$ ), we have  $R_{ik}R_{jk} \in G$  ( $i, j, k$  mutually different). Multiplying by  $R_{ij}^2$  from the left of a matrix adds twice of the  $j$ -th row to the  $i$ -th, and by  $R_{ik}R_{jk}$  adds the  $k$ -th row to both the  $i$ -th and the  $j$ -th, and by  $Q_{ij}$  switches the  $i$ -th row and  $j$ -th row up to a sign. We claim that any  $U \in \text{SL}(p, \mathbb{Z})$  with odd column sums becomes diagonal in  $\pm 1$ 's under finitely many such operations, by a Euclid type algorithm described below.

Suppose  $U = (u_{ij})_{1 \leq i, j \leq p}$ . In the first column there are oddly many odd entries, say  $u_{i_1, 1}, \dots, u_{i_{2m+1}, 1}$ . Adding the  $i_1$ -th row simultaneously to the  $i_2$ -th,  $\dots$ ,  $i_{2m+1}$ -th rows (i.e. multiplying  $U$  from the left by  $R_{i_2 i_1} R_{i_3 i_1} \dots R_{i_{2m+1} i_1}$ ) if necessary, we may assume only one entry is odd, and switching that row with the first row if necessary, we may further assume  $u_{11}$  is the only odd entry in the first column. Now there is some nonzero entry with a minimum absolute value, say  $u_{r1}$ . As  $U \in \text{SL}(p, \mathbb{Z})$ ,  $u_{r1}$  cannot divide all other entries in this column unless  $u_{r1} = \pm 1$ , so if  $u_{r1} \neq \pm 1$ , there must be some  $u_{s1}$  with  $|u_{s1}| > |u_{r1}|$  and  $u_{r1}$  cannot divide  $u_{s1}$ . By adding (or subtracting) an even times of the  $r$ -th row to the  $s$ -th, the  $u_{s1}$  becomes  $u'_{s1}$  such that  $-|u_{r1}| < u'_{s1} < |u_{r1}|$ . Because this operation does not change parity, we may repeat this process until we get a matrix with  $u_{11} = \pm 1$ , and other entries in the first column being even. Then add (or subtract) several even times of the first row to other rows, we obtain a matrix  $U''$  with the entries in the first column being zero except  $u_{11} = \pm 1$ . Apply the process recursively on the  $(p-1) \times (p-1)$ -submatrix  $U''_{11} = (u''_{ij})_{2 \leq i, j \leq p}$ , and use  $u_{22} = \pm 1$  to kill other even entries in the second column, and so on. In the end  $U$  becomes a diagonal matrix with  $\pm 1$ 's on the diagonal.

Finally, the claim means that any  $U$  with odd column sums can be written as  $U = KD$  where  $K$  is a word in  $R_{ij}^2$ ,  $Q_{ij}$ ,  $R_{ik}R_{jk}$ , and  $D$  is diagonal in  $\pm 1$ 's. Moreover,  $D$  must have even  $-1$ 's on the diagonal since the determinant is 1, for example, at the  $i_1, \dots, i_{2m}$  place,  $0 \leq 2m \leq p$ , then  $D = Q_{i_1 i_2}^2 \dots Q_{i_{2m-1} i_{2m}}^2$ . Therefore  $U = KD \in G$ .

To see any element  $U \in G$  has odd column sums, note that this condition is the same as saying that  $X\bar{U} = X$  for  $X = (1, \dots, 1) \in \mathbb{Z}_2^p$ , where  $\bar{U}$  is the modulo 2 reduction of  $U$ . As all the  $R_{1i}^2$ ,  $Q_{1i}$  ( $1 < i \leq p$ ) and  $R_{1p}R_{ip}$  ( $1 < i < p$ ) fix  $X$ , so does  $G$ . Therefore we see  $G$  consists of elements in  $\text{SL}(p, \mathbb{Z})$  with odd column sums. Furthermore, the index of  $G$  in  $\text{SL}(p, \mathbb{Z})$  is  $2^p - 1$  because  $\text{SL}(p, \mathbb{Z})$  acts transitively

on  $\mathbb{Z}_2^p - \{0\}$  modulo 2 and  $G$  is the stable subgroup at  $X$ .

(2) Instead of doing row operations, do the first step of the algorithm by column operations on the  $i$ -th row of  $U^{-1}$  to make the  $(i, r)$ -th entry  $\pm 1$ . Switch the  $i$ -th and  $r$ -th column, and multiply  $Q_{ir}^2$  if necessary to make the  $(i, i)$ -th entry  $+1$ . In other words, the  $(i, i)$ -th entry of  $U^{-1}K$  is 1 for some word  $K$  in  $R_{1i}^2, Q_{1i}$ . Then let  $J = (U^{-1}K)^{-1}$ .  $\square$

*Proof of Theorem 1.4.* By Lemma 2.8 and Lemma 2.9,  $E_i \supset G$ . Thus the index of  $E_i$  is bounded by the index of  $G$ , which is  $2^p - 1$  by Lemma 2.10 (1).  $\square$

### 3 Realizing hyperbolic attractors derived from expanding maps

We prove Theorem 1.2 in this section. To fix notation,  $u = (u_1, \dots, u_p)$  denotes a point in  $T^p = S_1^1 \times \dots \times S_p^1$  and  $(u, z)$  denotes a point in  $T^p \times D^2$ . Expanding maps and automorphisms on  $T^p$  are identified with their matrices, often written as  $\phi_A, \tau_U$ , etc. We also use  $T_i^{p-1}$  to denote  $S_1^1 \times \dots \times \hat{S}_i^1 \times \dots \times S_p^1$ .

Given an expanding map  $\phi : T^p \rightarrow T^p$ , we wish to use an unknotted embedding to realize an attractor derived from  $\phi$ . It is not hard to lift it as a hyperbolic bundle embedding first.

**Proposition 3.1.** *Let  $\phi : T^p \rightarrow T^p$  be any expanding map. There is a hyperbolic bundle embedding  $e : T^p \times D^2 \rightarrow T^p \times D^2$  lifted from  $\phi$ .*

*Proof.* Let  $A$  be the matrix of  $\phi$ . Recall that a Smith normal form ([Ne] Theorem II.9) of an integer matrix  $A = U\Delta V$  where  $U, V \in \text{SL}(p, \mathbb{Z})$  and  $\Delta = \text{diag}(\delta_1, \dots, \delta_p)$ , with  $\delta_i$  dividing  $\delta_{i+1}$  ( $1 \leq i \leq p-1$ ), and  $\delta_i > 0$  ( $1 \leq i \leq p$ ) here. Let  $\Delta_i = \text{diag}(1, \dots, \delta_i, \dots, 1)$ , then

$$A = U\Delta_1 \cdots \Delta_p V.$$

We first lift each factor to a bundle embedding  $e_U, e_V, e_{\Delta_i} : T^p \times D^2 \rightarrow T^p \times D^2$ .

To lift  $\tau_U$ , define, for example,  $e_U(u, z) = (\tau_U(u), u_1^{m_1} \cdots u_p^{m_p} z)$  for chosen integers  $m_1, \dots, m_p$ . Similarly lift  $\tau_V$  to  $e_V$ . To lift  $\phi_{\Delta_i}$ , first pick a hyperbolic bundle embedding  $b_{\delta_i} : S_i^1 \times D^2 \rightarrow S_i^1 \times D^2$  such that  $b_{\delta_i}(u_i, z) = (u_i^{\delta_i}, \bar{b}_i(u_i, z))$  sends the solid torus into itself as a connected thickened closed braid with winding number  $\delta_i$ , shrinking

evenly on the disk direction. Then define

$$e_{\Delta_i}(u, z) = (\phi_{\Delta_i}(u), \bar{b}_i(u_i, z)) = (b_{\delta_i} \times \text{id}_{T_i^{p-1}})(u, z).$$

Finally, take the composition  $e = e_U \circ e_{\Delta_1} \circ \cdots \circ e_{\Delta_p} \circ e_V$ , and we obtain a hyperbolic bundle embedding lifted from  $\phi$ .  $\square$

Although the lifting is never unique, we have to pick a topologically simple one, after which a  $T^p$  in  $\mathbb{R}^{p+2}$  will not get ‘knottier’ or ‘more twisted’, so as to realize its attractor in  $\mathbb{R}^{p+2}$ .

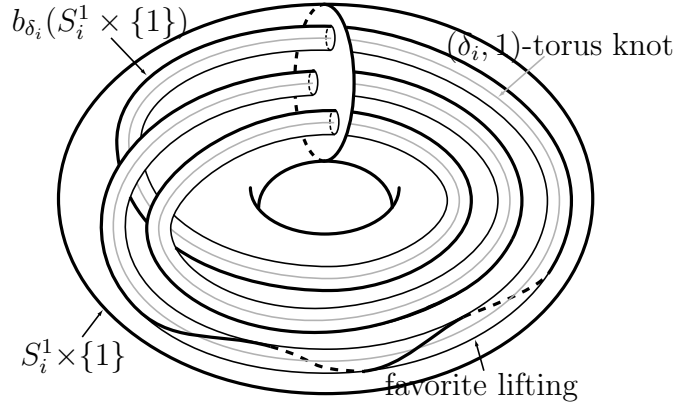


Figure 2. The favorite lifting and untwisted framing, where  $\delta_i = 3$

**Example 3.2.** [The favorite lifting] Define  $e_U(u, z) = (\tau_U(u), z)$ ,  $e_V(u, z) = (\tau_V(u), z)$ , and  $e_{\Delta_i}(u, z) = b_{\delta_i} \times \text{id}_{T_i^{p-1}}$ , with

$$b_{\delta_i}(u_i, z) = (u_i^{\delta_i}, \frac{1}{2}u_i + \frac{1}{\delta_i^2}u_i^{1-\delta_i}z).$$

This specific choice of  $b_{\delta_i}$  is presented in Figure 2, the image  $b_{\delta_i}(S_i^1 \times \{0\})$  is the  $(\delta_i, 1)$  cable in the solid torus  $S_i^1 \times D^2$ , with  $\delta_i = 3$ .

This specific choice of  $b_{\delta_i}$  gives us a favorite lifting  $e = e_U \circ e_{\Delta_1} \circ \cdots \circ e_{\Delta_p} \circ e_V$  of  $\phi$ .

An embedding  $j : T^p \times D^2 \rightarrow \mathbb{R}^{p+2}$  is, in general, understood by knowing its core restriction  $\iota = j|_{T^p \times \{0\}}$ , and the framing. We need the following definition.

**Definition 3.3.**  $j : T^p \times D^2 \rightarrow \mathbb{R}^{p+2}$  is called *unknotted*, if its core restriction  $\iota$  is unknotted. The *type* of  $j$  is the type of  $\iota$ . The embedding  $j$  is called with *untwisted framing*, if  $j(T^p \times \{1\})$  is null-homologous in the complement of the core  $\iota(T^p)$ .

**Proposition 3.4.** *Unknotted embeddings with untwisted framing are unique of its type, up to  $\mathbb{R}^{p+2}$  self-diffeomorphism. Namely for unknotted embeddings  $j_0, j_1$  with untwisted framing of the same type, there is a  $\mathbb{R}^{p+2}$  self-diffeomorphism  $h : \mathbb{R}^{p+2} \rightarrow \mathbb{R}^{p+2}$  such that  $h \circ j_0 = j_1$ .*

*Proof.* By definition we may first find a  $\mathbb{R}^{p+2}$  self-diffeomorphism  $h$  so that  $\iota_1 = h \circ \iota_0$ , and in the smooth category one may assume  $h$  preserves the bundle structures, namely,

$$j_1(u, z) = h(j_0(u, u_1^{m_1} \cdots u_p^{m_p} z))$$

for  $u \in T^p$  and  $m_1, \dots, m_p \in \mathbb{Z}$ . Because  $j_1(u, 1) = h(j_0(u, u_1^{m_1} \cdots u_p^{m_p}))$  is a null-homologous  $p$ -torus in  $\mathbb{R}^{p+2} - \iota_1(T^p)$  by the untwisted-framing condition, and  $h(j_0(u, 1))$  is null-homologous in  $\mathbb{R}^{p+2} - h(\iota_0(T^p))$  by the untwisted-framing condition, and  $\iota_1(T^p) = h(\iota_0(T^p))$ , with  $H_p(\mathbb{R}^{p+2} - \iota_1(T^p)) \cong \mathbb{Z}^p$  understood, we conclude  $u_1^{m_1} \cdots u_p^{m_p} = 1$ , so  $m_i = 0$  for  $1 \leq i \leq p$ . Therefore  $j_1 = h \circ j_0$ .  $\square$

This basically says a topologically ‘simple’ embedding  $j$  is determined by its type. Fortunately our favorite lifting  $e$  is topologically simple, namely it preserves the untwisted-framing property, unknotted-ness, and modularity. We will prove in Lemma 3.6 and Lemma 3.8, and also remark that this should be viewed as a generalization of the following well-known fact in the classical knot theory that  $b_{\delta_i}$  is topologically simple.

**Lemma 3.5.** (1)  $b_{\delta_i}(S_i^1 \times \{1\})$  is homological to  $\delta_i$  times  $S_i^1 \times \{1\}$  in  $S_i^1 \times D^2 - b_{\delta_i}(S_i^1 \times \text{Int}(D^2))$  (see Figure 2). Hence if  $j : S_i^1 \times D^2 \rightarrow \mathbb{R}^3$  has untwisted framing, then  $j \circ b_{\delta_i}$  has untwisted framing.

(2) Furthermore if  $j$  in (1) is also unknotted, then  $b_{\delta_i}(S_i^1 \times \{0\})$  is the  $(\delta_i, 1)$  torus knot, which is unknotted.

**Lemma 3.6.** If  $j : T^p \times D^2 \rightarrow \mathbb{R}^{p+2}$  has untwisted framing, then  $j \circ e$  also has untwisted framing.

*Proof.* Obviously the composition with  $e_U, e_V$  preserves untwisted framing as they are identity on the fiber  $D^2$ . We claim  $e_{\Delta_1}, \dots, e_{\Delta_p}$  preserves untwisted framing, then provided that  $j$  has untwisted framing, so does  $j \circ e_U$ , and hence so does  $(j \circ e_U) \circ e_{\Delta_1}$ , and so on. Finally  $j \circ e$  also has untwisted framing.

Since  $e_{\Delta_i} = b_{\delta_i} \times \text{id}_{T_i^{p-1}} : S_i^1 \times D^2 \times T_i^{p-1} \rightarrow S_i^1 \times D^2 \times T_i^{p-1}$ , by Lemma 3.5 (1), we have  $e_{\Delta_i}(T^p \times \{1\})$  is homological to  $\delta_i$  times  $T^p \times \{1\}$  in  $T^p \times D^2 - e_{\Delta_i}(T^p \times \text{Int}(D^2))$ .

Thus if  $j : T^p \times D^2 \rightarrow \mathbb{R}^{p+2}$  has untwisted framing, then  $j \circ e_{\Delta_i}$  has untwisted framing.  $\square$

The claim that  $e$  also preserves the unknotted-ness, namely  $j \circ e$  remains unknotted provided  $j$  is unknotted, is much less obvious. Before we explain in general cases, we investigate an essential case when  $e$  is just  $e_{\Delta_i}$ , and  $j$  is a special candidate of its type.

We will call  $j_p : T^p \times D^2 \rightarrow \mathbb{R}^{p+2}$  *standard*, if  $j_p$  has untwisted framing, and  $j_p|_{T^p \times \{0\}} = \iota_p$ .

To help visualize how  $j \circ e(T^p \times D^2)$  unknots itself, note by Corollary 2.6, a standard type  $j_p$  can be written as  $k_i \circ g_i$ , for  $i = 1, \dots, p$ , where

$$g_i : S_1^1 \times \dots \times S_p^1 \times D^2 \subset (S_1^1 \times \dots \times \hat{S}_i^1 \times \dots \times S_p^1) \times D^3$$

is the identity on factors  $S_j^1$ ,  $j \neq i$ , and  $S_i^1 \times D^2 \subset D^3$  is the thicken-up of the circle lying on the equatorial disc of  $D^3$ , centered at the origin with radius one half.

**Lemma 3.7.** *Suppose  $1 \leq i \leq p$ , and  $j_J = j_p \circ e_J$ , where  $J \in \text{SL}(p, \mathbb{Z})$  satisfies the minor  $J_{ii}^* = 1$ . Then  $j_J \circ e_{\Delta_i}$  is also unknotted. Moreover, the type of  $j_J \circ e_{\Delta_i}$  is modular.*

*Proof.* We only need to prove  $i = 1$  for example. Denote  $g_J = g_1 \circ e_J$ . For the reader's reference, all maps involved here are shown in the commutative diagram below:

$$\begin{array}{ccccc}
 & & T_1^{p-1} \times D^3 & & \\
 & \nearrow g_J & \uparrow g_1 & \searrow k_1 & \\
 T^p \times D^2 & \xrightarrow{e_{\Delta_1}} & T^p \times D^2 & \xrightarrow{e_J} & T^p \times D^2 \xrightarrow{j_p} \mathbb{R}^{p+2} \\
 \downarrow \pi & & \downarrow \pi & & \downarrow \pi \\
 T^p & \xrightarrow{\phi_{\Delta_1}} & T^p & \xrightarrow{\tau_J} & T^p
 \end{array}$$

There is a basic picture for  $p = 2$ , namely torus in  $\mathbb{R}^4$ , to keep in mind. Let  $K'$  be a  $(\delta_1, 1)$ -type torus knot, of which the torus lies parallel to the  $xy$ -plane. It is of course unknotted. Imagine an  $S^1$ -family of  $K'_w$  in  $D^3$ , such that as  $w$  goes around  $S^1$ , the  $K'_w$  rotates around the  $z$ -axis several times. We may simultaneously cap off all these knots by picking a disk with boundary being  $K'_1$  and rotating it along with  $K'_w$ , as  $w$  goes around  $S^1$ . See Figure 3. This implies that  $\bigcup_{w \in S^1} K'_w$  is an unknotted torus in  $S^1 \times D^3$  (in the sense that it is diffeotopic to  $S^1 \times S^1 \subset S^1 \times D^3$ ). Therefore, if  $S^1 \times D^3$  is at first embedded in  $\mathbb{R}^4$ , the image of  $\bigcup_{w \in S^1} K'_w$  is also an unknotted torus in  $\mathbb{R}^4$ .

For the case  $p = 2$ , the fact that the type is changed by a modular transformation is trivially true because the smooth mapping class group  $\pi_0 \text{Diff}_+(T^2) \cong \text{SL}(2, \mathbb{Z})$ ; (for  $p > 2$  we need to carefully choose our construction).

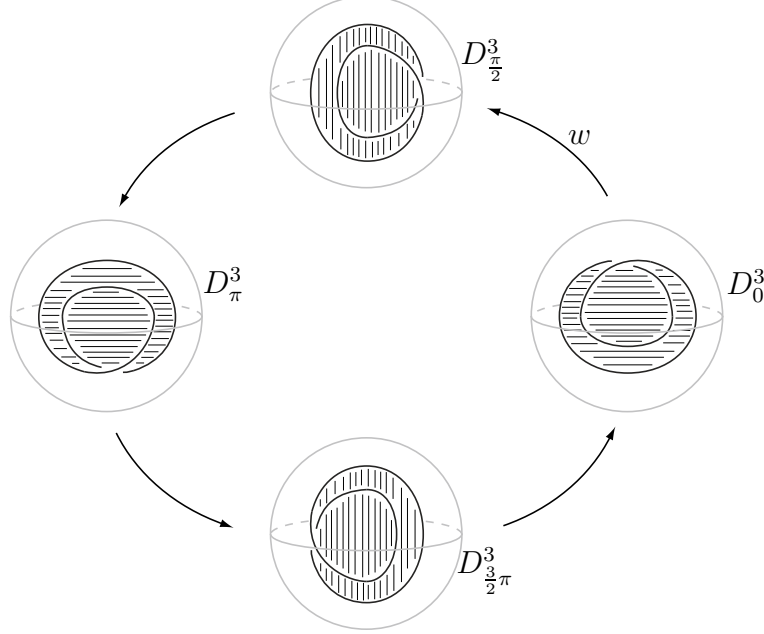


Figure 3.  $S^1$ -family of rotating  $(\delta_1, 1)$  torus knots with disks capped, where  $\delta_1 = 2$

As one would figure out later,  $\bigcup_{w \in S^1} K'_w$  is exactly  $g_{R_{12}} \circ e_{\Delta_1}(T^2 \times \{0\})$ . In general, let us explain how the proof of the lemma essentially leads to a picture like this, except now with the  $T_1^{p-1}$ -family instead of just with the  $S^1$ -family.

The idea to prove the lemma is that we wish to diffeotope the core  $K' = g_J \circ e_{\Delta_1}(T^p \times \{0\})$  back to  $K = g_1(T^p \times \{0\})$  within  $T_1^{p-1} \times D^3$ , before including the latter into  $\mathbb{R}^{p+2}$  via  $k_1$ . The torus  $K' \subset T_1^{p-1} \times D^3$  may be viewed as a  $T_1^{p-1}$ -family of (hopefully) unknotted loops in  $D^3$ . Intuitively  $S_2^1, \dots, S_p^1$  are independent clocks, and at every ‘moment’  $(u_2, \dots, u_p)$  we see a loop in  $D^3$ . It turns out that this is a  $(\delta_1, 1)$  torus knot rotating around a fixed axis, an analogue of the basic picture we mentioned above. Then we may simultaneously diffeotope the loops back to the standard place in the  $D^3$  fibers and we are done. Therefore the key is to read out this picture by understanding the intersection loci of  $K'$  on fibers, namely  $K'|_{(u_2, \dots, u_p)} = K' \cap (\{(u_2, \dots, u_p)\} \times D^3)$ .

Denote  $v = \phi(u) = \tau_J \circ \phi_{\Delta_1}(u)$ , and  $\vec{\omega}(u_1) = u_{1x} \cdot \vec{e}_1 + u_{1y} \cdot \vec{e}_2$  a rotating vector in

$\mathbb{R}_{1,2}^2$ . Then

$$g_1(u, z) = ((u_2, \dots, u_p), \frac{1}{2} \cdot \vec{\omega}(u_1) + \frac{1}{3}[z_x \cdot \vec{\omega}(u_1) + z_y \cdot \vec{\varepsilon}_3]).$$

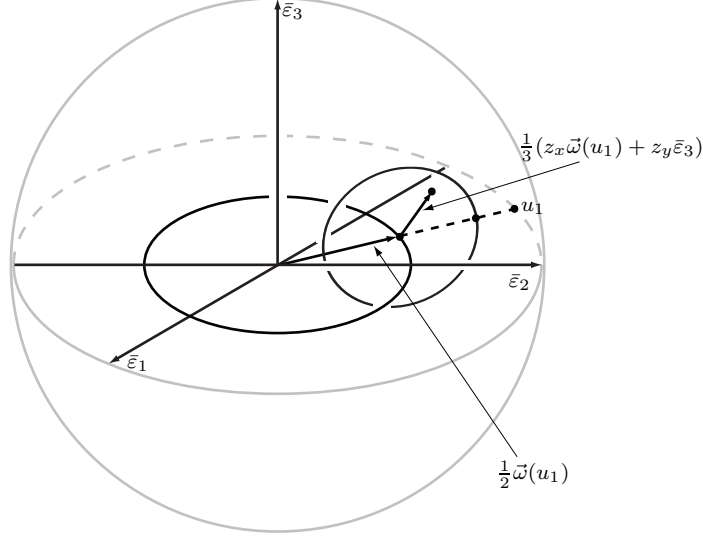


Figure 4. image of  $(u_1, z)$  under  $g_1 : S_1^1 \times D^2 \rightarrow D^3$

Composing  $e_J(u, z) = (\tau_J(u), z)$  with  $e_{\Delta_1}(u, z) = (\phi_{\Delta_1}(u), \frac{1}{2}u_1 + \frac{1}{\delta_1^2}u_1^{1-\delta_1}z)$ , we have

$$g_J \circ e_{\Delta_1}(u, 0) = ((v_2, \dots, v_p), b(u)),$$

where  $v_j$  means the  $S_j^1$ -component of  $v = \phi(u)$ , and

$$b(u) = \frac{1}{2} \cdot \vec{\omega}(v_1) + \frac{1}{6}[u_{1x} \cdot \vec{\omega}(v_1) + u_{1y} \cdot \vec{\varepsilon}_3].$$

Because  $J_{11}^* = 1$  is the  $(1, 1)$ -th entry of  $J^{-1}$ ,  $u_1^{\delta_1} = v_1 v_2^{-r_2} \dots v_p^{-r_p}$  for some integers  $r_2, \dots, r_p$ , so  $v_1 = u_1^{\delta_1} v_2^{r_2} \dots v_p^{r_p}$ . Therefore at the ‘moment’  $(v_2, \dots, v_p) \in T_1^{p-1}$ ,  $K'|_{(v_2, \dots, v_p)}$  is a  $(\delta_1, 1)$ -torus knot in  $D^3$  defined by  $b(u)$ , and as  $v_j = e^{i\theta_j}$  goes along  $S_j^1$ ,  $K'|_{(v_2, \dots, v_p)}$  rotates about the  $\vec{\varepsilon}_3$ -axis by an angle  $r_j \theta_j$ .

Note that a  $(\delta_1, 1)$ -torus knot in  $D^3$  is unknotted, so there is a diffeotopy  $h_t : D^3 \rightarrow D^3$  with the identity near the boundary, such that  $h_0 = \text{id}_{D^3}$  and  $h_1(K'|_{(1, \dots, 1)}) = K|_{(1, \dots, 1)}$ . To unknot  $K'$  simultaneously on fibers, define  $\rho : T_1^{p-1} \rightarrow \text{SO}(3)$ , with  $\rho(v_2, \dots, v_p)$  being the rotation about the  $\vec{\varepsilon}_3$ -axis by an angle  $r_2 \theta_2 + \dots + r_p \theta_p$ , where  $v_j = e^{i\theta_j}$ . The ‘unknotting’ diffeotopy may be defined as

$$H_t : T_1^{p-1} \times D^3 \rightarrow T_1^{p-1} \times D^3$$



with

$$H_t((v_2, \dots, v_p), \vec{x}) = ((v_2, \dots, v_p), \rho(v_2, \dots, v_p) \circ h_t \circ (\rho(v_2, \dots, v_p))^{-1}(\vec{x})).$$

Since  $H_t$  is the identity near the boundary, when we embed  $T_1^{p-1} \times D^3$  into  $\mathbb{R}^{p+2}$  by  $k_1$ ,  $H_t$  induces a diffeotopy identity near the boundary on  $k_1(T_1^{p-1} \times D^3)$ , which extends as a diffeotopy of  $\mathbb{R}^{p+2}$ . This ambient diffeotopy changes  $j_J \circ e_{\Delta_1}(T^p \times \{0\})$  back to  $j_p(T^p \times \{0\})$ , so the former is as unknotted as the latter.

To see the moreover part, note that both  $g_1, H_1 \circ g_J \circ e_{\Delta_1}$  embed  $T^p \times \{0\}$  into  $T_1^{p-1} \times D^3$  with the same image. Let  $B$  denote the matrix obtained from  $J$  by multiplying each entry of the first column by  $\delta_1$  and then changing the first row to  $(1, 0, \dots, 0)$ . Since  $J_{11}^* = 1$ ,  $B \in \text{SL}(p, \mathbb{Z})$ . By comparing

$$H_1 \circ g_J \circ e_{\Delta_1}(u, 0) = (v_2, \dots, v_p, \rho(v_2, \dots, v_p) \circ h_1 \circ (\rho(v_2, \dots, v_p))^{-1} \circ b(u))$$

with

$$g_1 \circ e_B(u, 0) = (v_2, \dots, v_p, \frac{1}{2}\vec{\omega}(u_1)),$$

we see that the self-diffeomorphism  $(H_1 \circ g_J \circ e_{\Delta_1}) \circ (g_1 \circ e_B)^{-1} : T^p \rightarrow T^p$  can be written as  $F : T_1^{p-1} \times S^1 \rightarrow T_1^{p-1} \times S^1$ ,  $F((v_2, \dots, v_p), z) = ((v_2, \dots, v_p), f(v_2, \dots, v_p)(z))$ , where  $f : T_1^{p-1} \rightarrow \text{Diff}_+(S^1)$ . Because  $\text{Diff}_+(S^1) \simeq SO(2)$ , we conclude that  $f$  is homotopic to  $f_0 : T_1^{p-1} \rightarrow SO(2)$ , where  $f_0(v_2, \dots, v_p) = v_2^{m_2} \cdots v_p^{m_p}$  for some integers  $m_2, \dots, m_p$ , and  $F$  is diffeotopic to the (linear) automorphism  $F_0 : T_1^{p-1} \times S^1 \rightarrow T_1^{p-1} \times S^1$ ,  $F_0((v_2, \dots, v_p), z) = ((v_2, \dots, v_p), v_2^{m_2} \cdots v_p^{m_p} z)$ . Let  $M$  denote the matrix obtained from the  $p \times p$  identity matrix by changing the first row to  $(1, m_2, \dots, m_p)$ . Then the type of  $j_J \circ e_{\Delta_1}$  is a modular transformation of the standard type by  $M \circ B$ .  $\square$

The general case that  $e$  preserves unknotted-ness and modularity is as follows.

**Lemma 3.8.** *If  $j$  is unknotted of modular type with untwisted framing, then  $j \circ e : T^p \times D^2 \rightarrow \mathbb{R}^{p+2}$  is also unknotted of modular type.*

*Proof.* Clearly  $\hat{j} = j \circ e_U$  is unknotted with modular type and untwisted framing. Note  $\hat{j} \circ e_{\Delta_1}$  is unknotted if and only if so is  $h \circ \hat{j} \circ e_{\Delta_1}$  for any  $\mathbb{R}^{p+2}$  self-diffeomorphism  $h$ . By Proposition 3.4, the unknotted-ness of  $\hat{j} \circ e_{\Delta_1}$  depends only on the type of  $\hat{j}$ . Since  $\hat{j}$  is of modular type, suppose  $j_J = j_p \circ e_J$  has the same type as  $\hat{j}$ . Moreover,  $J$  may be picked so that  $J_{11}^* = 1$  by Lemma 2.10(2). Then by Lemma 3.7,  $j_J \circ e_{\Delta_1}$

remains unknotted with modular type. Therefore  $\hat{j} \circ e_{\Delta_1}$  is unknotted with modular type. By Lemma 3.6,  $\hat{j} \circ e_{\Delta_1}$  has untwisted framing.

Repeat this argument so  $(\hat{j} \circ e_{\Delta_1}) \circ e_{\Delta_2}$  is unknotted with modular type and untwisted framing, and so on we see that  $\hat{j} \circ e_{\Delta_1} \cdots e_{\Delta_p}$  is also unknotted with modular type. Finally  $(j \circ e_U \circ e_{\Delta_1} \circ \cdots \circ e_{\Delta_p}) \circ e_V$  is also unknotted since it has the same image of the core as  $j \circ e_U \circ e_{\Delta_1} \circ \cdots \circ e_{\Delta_p}$ , and the type change is modular. We conclude that  $j \circ e$  is still unknotted of modular type.  $\square$

*Proof of Theorem 1.2.* Pick an unknotted embedding  $j : T^p \times D^2 \rightarrow \mathbb{R}^{p+2}$  of modular type with untwisted framing. (We can in fact require the core image to be any unknotted  $T^p$  in  $\mathbb{R}^{p+2}$ , by Proposition 2.4.) By Lemma 3.6 and Lemma 3.8,  $j \circ e^i$  ( $i \geq 0$ ) are all unknotted of modular type with untwisted framing. By Theorem 1.4, Definition 2.2 and its remark, at least two of  $j \circ e^i$  ( $0 \leq i \leq 2^p - 1$ ) are of the same type. Suppose  $j \circ e^k$  and  $j \circ e^l$  ( $0 \leq k < l \leq 2^p - 1$ ) are of the same type. Pick the embedding  $\hat{j} = j \circ e^k : T^p \times D^2 \rightarrow \mathbb{R}^{p+2}$  instead of  $j$ , and let  $d = l - k$ . There is an  $\mathbb{R}^{p+2}$  self-diffeomorphism  $h$  such that  $h \circ \hat{j} = \hat{j} \circ e^d$  by Proposition 3.4. This is to say,  $e^d$  can be realized by an embedding  $\hat{j} : T^p \times D^2 \rightarrow \mathbb{R}^{p+2}$  with extension  $h$ .

Therefore,

$$\Lambda = \bigcap_{i=0}^{\infty} e^i(T^p \times D^2) = \bigcap_{i=0}^{\infty} e^{di}(T^p \times D^2)$$

embeds into  $\mathbb{R}^{p+2}$  by  $\hat{j}$  as an attractor of  $h$ , so we have realized an expanding attractor derived from  $\phi$ .  $\square$

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## References

- [B] H. Bothe, *The ambient structure of expanding attractors. II. Solenoids in 3-manifolds*. Math. Nachr. 112 (1983), 69–102.
- [DLWY] F. Ding, Y. Liu, S. Wang, J. Yao, *A spin obstruction for codimension-two diffeomorphism extension*. preprint, 2009.
- [DPWY] F. Ding, J. Pan, S. Wang, J. Yao, *Manifolds with  $\Omega(f)$  a union of DE attractors are rational homology spheres*. arXiv:0812.1260

- [Gr] M. Gromov, *Groups of polynomial growth and expanding maps*. Inst. Hautes Études Sci. Publ. Math. No. 53 (1981), 53–73.
- [JNW] B. Jiang, Y. Ni, S. Wang, *3-manifolds that admit knotted solenoids as attractors*. Trans. Amer. Math. Soc. 356 (2004), no. 11, 4371–4382.
- [JWZ] B. Jiang, S. Wang, H. Zheng, *No embeddings of solenoids into surfaces*. Proc. Amer. Math. Soc. 136 (2008), no. 10, 3697–3700.
- [Mo] J.M. Montesinos, *On twins in the four-sphere. I*. Quart. J. Math. Oxford Ser. (2) 34 (1983), no. 134, 171–199.
- [Ne] M. Newman, *Integral matrices*, Academic Press, 1972.
- [Sm] S. Smale, *Differentiable dynamical systems*, Bull. Amer. Math. Soc. 73 (1967), 747–817.

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